

# Pedestals: Polynomial matrices with polynomial eigenvalues

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joint work with

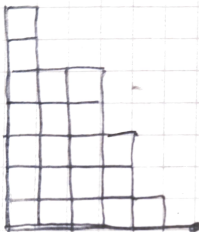
Richard Kenyon, Maxim Konsevich, Oleg Ogievetsky,  
Cosmin Pohoata, Will Sawin

# Young diagrams

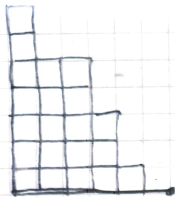
Let us write the integer  $k$  as follows:

$$k = \pi(1) + \pi(2) + \dots + \pi(n),$$

with  $\pi(i) \geq 0$ ,  $\pi(i) \geq \pi(i+1)$ . We call this function  $\pi$  a *partition of  $k$  into (at most)  $n$  parts*. Let  $\mathcal{Y}_n$  denote the set of all partitions  $\pi$ -s of arbitrary integers. They are called Young diagrams with at most  $n$  columns. We call  $k$  to be the volume of the diagram  $\pi$ .



1 2 3 4 5 6



1 2 3 4 5 6

$$21 = 7 + 5 + 5 + 3 + 1 + 0$$

Let  $g_k$  be the number of partitions  $\pi$  of  $k$ . The generation function of the sequence  $g_k$  is given by:

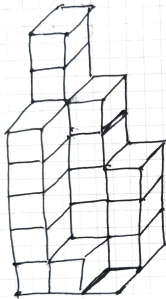
$$G_n(t) = \prod_{l=1}^n \frac{1}{1-t^l}.$$

Indeed,

$$\prod_{l=1}^n \frac{1}{1-t^l} =$$
$$= (1+t+t^2+t^3+\dots)(1+t^2+t^4+\dots)(1+t^3+t^6+\dots)\dots$$
$$(1+t^n+t^{2n}+t^{3n}+\dots).$$

# 3D Young diagrams (Plane partitions)

Consider now the plane partitions, sitting over rectangle  $n \times m$ .



7 5 3  
5 1 0



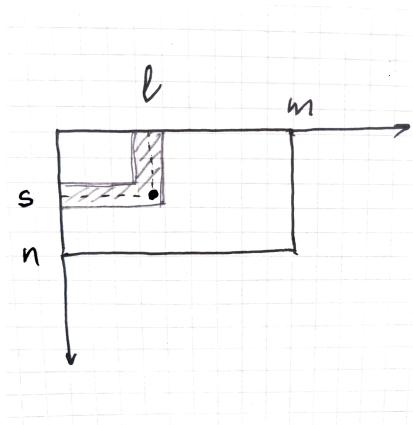
## 3D Young diagrams (Plane partitions)

Let now  $g_k$  be the number of plane partitions of volume  $k$ , sitting over rectangle  $n \times m$ . The generation function of the sequence  $g_k$  is given by the Mac-Mahon formula:

$$G_{n \times m}(t) = \prod_{l=1}^n \prod_{s=1}^m \frac{1}{1 - t^{l+s-1}}.$$

(The sum  $l + s - 1$  is the hook length of the cell  $(l, s)$ .)

# The hook length

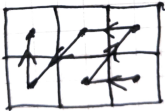
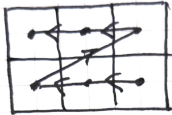


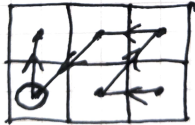
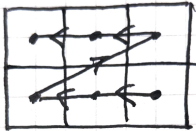
There is a natural map from the set  $\Pi_{n,m}$  of plane partitions sitting over rectangle  $n \times m$ , onto the Young diagrams  $\mathcal{Y}_{nm}$ .

There is a natural map from the set  $\mathcal{P}_{n,m}$  of plane partitions sitting over rectangle  $n \times m$ , onto the Young diagrams  $\mathcal{Y}_{nm}$ .  
How about the inverse map?

The rectangle  $n \times m$  has its natural partial order. Let us fix some linear order  $P$  on it, which extends the partial order. It is just a map from the rectangle  $n \times m$  onto the segment  $[1, nm]$ . And let  $Q$  be any other linear order on it.

We call the node  $Q^{-1}(k)$  a  $(P, Q)$ -disagreement node iff  $P(Q^{-1}(k-1)) > P(Q^{-1}(k))$ .





Given the pair  $P, Q$ , we define the function  $q_{PQ}$  on the rectangle  $n \times m$  by

$$\begin{aligned} q_{PQ}(Q^{-1}(k)) &= \\ &= \# \{l : l \leq k, Q^{-1}(l) \text{ is a } (P, Q)\text{-disagreement node}\}. \end{aligned}$$

Clearly, the function  $q_{PQ}$  is non-decreasing on the rectangle. It is called the pedestal of  $Q$  with respect to  $P$ . Let  $\mathcal{E}_P$  denotes the set of all pedestals  $q_{PQ}$ .



P



Q



$g^{PQ}$

1	0	0
1	0	0

## Theorem

*There is a bijection between the set  $\mathcal{P}_{n,m}$  of nondecreasing functions (i.e. 3D diagrams) and the direct product  $\mathcal{E}_P \times \mathcal{Y}_{nm}$ , respecting the volumes.*

It is given by the following construction: to each pedestal  $q_{PQ}$  and each partition  $\pi$  (i.e. 2D diagram) it corresponds the following function  $p$  on  $n \times m$  :

$$p(Q^{-1}(k)) = q_{PQ}(Q^{-1}(k)) + \pi(k), \quad k = 1, \dots, mn.$$

Clearly, the function thus defined is non-decreasing on the rectangle  $n \times m$ .

Therefore we have to fix some ordering  $P$  on the rectangle  $n \times m$ , consider all the pedestals  $q_{PQ}$ , and take the generating function

$$\Pi_P(t) = \sum_Q t^{v(q_{PQ})}$$

(in fact, generating polynomial) of the sequence of the number of pedestals with a given volume. Then we have the identity:

$$G_{n \times m}(t) = \Pi_P(t) G_{nm}(t) \equiv \Pi_P(t) \prod_{l=1}^{nm} \frac{1}{1-t^l}.$$

In particular, the polynomial  $\Pi_P(t)$  does not depend on  $P$ , and thus can be denoted by  $\Pi_{n \times m}(t)$ .

In particular, we have

$$\begin{aligned}\prod_{n \times m}(t) &= \\ &= \frac{\prod_{l=1}^{nm} (1 - t^l)}{\prod_{l=1}^n \prod_{s=1}^m (1 - t^{l+s-1})},\end{aligned}$$

i.e. we see fine cancellations here.

# Partition (3,2)

The standard tableaux are

1	3	5
2	4	

1	2	5
3	4	

1	3	4
2	5	

1	2	4
3	5	

1	2	3
4	5	

# Pedestal matrix

$$\begin{pmatrix} 1 & q^3 & q & q^4 & q^2 \\ q^3 & 1 & q^4 & q & q^2 \\ q & q^4 & 1 & q^3 & q^2 \\ q^4 & q & q^3 & 1 & q^2 \\ q^4 & q & q^3 & q^2 & 1 \end{pmatrix}$$

$$\begin{aligned} & -(-1 + q)(1 + q) \\ & (-1 + q)^2 (1 + q + q^2) \\ & -(-1 + q)(1 + q) (1 + q + q^2) \\ & -(-1 + q)(1 + q) (1 - q + q^2) \end{aligned}$$

The fact that the function  $\Pi_P(t)$  does not depend on the order  $P$  on our rectangle has the following generalization. Instead of characterizing the pedestal  $q_{PQ}$  just by its volume let us associate with it the monomial

$$m_{PQ}(x_1, x_2, x_3, \dots) = x_1^{l_1-1} x_2^{l_2-l_1} \dots x_r^{l_r-l_{r-1}} x_{r+1}^{n-l_r+1},$$

where  $r$  is the number of  $(P, Q)$ -disagreement nodes, and  $l_1, \dots, l_r$  are their locations. Note that  $m_{PQ}(1, t, t^2, \dots) = t^{v(q_{PQ})}$ .



We have shown with Oleg Ogievetsky that the polynomial

$$\mathfrak{h}_P(x_1, x_2, x_3, \dots) = \sum_Q m_{PQ}(x_1, x_2, x_3, \dots)$$

is also independent of  $P$ , so it can be denoted as  $\mathfrak{h}_{n \times m}(x_1, x_2, x_3, \dots)$ .

We will go to the general case. Instead of the partially ordered set – the rectangle  $n \times m$  – we will consider any finite poset  $X$ . We will denote by  $\text{Tot}_X$  the set of all possible linear orders on  $X$ .

One way of expressing the property that  $\eta_P$ ,  $P \in \text{Tot}_X$  depends only on  $X$  is to say that the matrix  $M_X$  of size  $|\text{Tot}_X| \times |\text{Tot}_X|$ , with entries  $(M_X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \dots)$  is *stochastic*, i.e. the vector  $(1, 1, \dots, 1)$  is the right eigenvector, with the eigenvalue  $\eta_X(x_1, x_2, x_3, \dots)$ . The matrix  $M_X$  is the **pedestal matrix**.

## Theorem

*(Richard Kenyon, Maxim Konsevich, Oleg Ogievetsky, Cosmin Pohoata, Will Sawin, S.S.)*

*For every poset  $X$ , all the eigenvalues of the  $|\text{Tot}_X| \times |\text{Tot}_X|$ -matrix  $M_X$  with entries  $(M_X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \dots)$  are polynomials in  $x_1, x_2, x_3, \dots$  with integer coefficients.*

# Pedestal matrices - Example

Partition (3,2,1):

□□□

□□

□

The 16 standard tableaux (i.e. the orders  $P, Q$  on our Young tableau) are

$\{1, 4, 6, 2, 5, 3\}$  ,  $\{1, 3, 6, 2, 5, 4\}$  ,  $\{1, 2, 6, 3, 5, 4\}$  ,  $\{1, 3, 6, 2, 4, 5\}$  ,  
 $\{1, 2, 6, 3, 4, 5\}$  ,  $\{1, 4, 5, 2, 6, 3\}$  ,  $\{1, 3, 5, 2, 6, 4\}$  ,  $\{1, 2, 5, 3, 6, 4\}$  ,  
 $\{1, 3, 4, 2, 6, 5\}$  ,  $\{1, 2, 4, 3, 6, 5\}$  ,  $\{1, 2, 3, 4, 6, 5\}$  ,  $\{1, 3, 5, 2, 4, 6\}$  ,  
 $\{1, 2, 5, 3, 4, 6\}$  ,  $\{1, 3, 4, 2, 5, 6\}$  ,  $\{1, 2, 4, 3, 5, 6\}$  ,  $\{1, 2, 3, 4, 5, 6\}$  .

To save the space we write down the pedestal matrix in which the replacement

$$(x_1^6, x_1^5 x_2, x_1^4 x_2^2, x_1^4 x_2 x_3, x_1^3 x_2^3, x_1^3 x_2^2 x_3, x_1^2 x_2^4, x_1^2 x_2^3 x_3, x_1^2 x_2^2 x_3^2, x_1^2 x_2^2 x_3 x_4) \rightarrow \\ (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$$

is made.

# The matrix

$$\begin{pmatrix} a_1 & a_5 & a_7 & a_3 & a_9 & a_2 & a_6 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\ a_5 & a_1 & a_7 & a_3 & a_9 & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\ a_5 & a_7 & a_1 & a_9 & a_3 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_8 & a_2 & a_{10} & a_4 & a_6 \\ a_5 & a_3 & a_9 & a_1 & a_7 & a_6 & a_2 & a_8 & a_4 & a_{10} & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 \\ a_5 & a_9 & a_3 & a_7 & a_1 & a_6 & a_8 & a_2 & a_{10} & a_4 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 \\ a_2 & a_6 & a_8 & a_3 & a_9 & a_1 & a_5 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\ a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_1 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\ a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_7 & a_1 & a_9 & a_3 & a_5 & a_{10} & a_4 & a_8 & a_2 & a_6 \\ a_6 & a_2 & a_8 & a_4 & a_{10} & a_5 & a_3 & a_9 & a_1 & a_7 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 \\ a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_1 & a_5 & a_9 & a_3 & a_8 & a_2 & a_6 \\ a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_5 & a_1 & a_9 & a_3 & a_8 & a_6 & a_2 \\ a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_4 & a_{10} & a_2 & a_8 & a_6 & a_1 & a_7 & a_3 & a_9 & a_5 \\ a_5 & a_9 & a_3 & a_8 & a_2 & a_6 & a_{10} & a_4 & a_8 & a_2 & a_6 & a_7 & a_1 & a_9 & a_3 & a_5 \\ a_6 & a_4 & a_{10} & a_2 & a_8 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_3 & a_9 & a_1 & a_7 & a_5 \\ a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_9 & a_2 & a_6 & a_9 & a_3 & a_7 & a_1 & a_5 \\ a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_8 & a_6 & a_2 & a_9 & a_3 & a_7 & a_5 & a_1 \end{pmatrix}$$

# The eigenvalues

$$(a_1 - a_4 - a_7 + a_{10})^3, a_1 - a_4 + a_7 - a_{10}, (a_1 + a_2 - a_5 - a_6)^2, (a_1 - a_2 - a_5 + a_6)$$

$$(a_1 - a_2 - a_3 + a_4 + a_7 - a_8 - a_9 + a_{10})^2, (a_1 - a_2 - a_3 + a_4 - a_7 + a_8 + a_9 - a_{10})^2,$$

$$(a_1 - a_4 + a_5 - a_6 + a_7 - a_{10})^2, a_1 + 2a_2 + 2a_3 + a_4 - a_7 - 2a_8 - 2a_9 - a_{10},$$

$$a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + a_7 + 2a_8 + 2a_9 + a_{10}.$$

The plan of the proof :

- 1 We introduce the class of certain matrices  $M_F$ , such that the matrix  $M_X$  can be written as a linear combination of  $M_F$ -s with integer coefficients.
- 2 We show that all  $M_F$ -s can be made upper-triangular via the conjugation with the **same** matrix, and the resulting upper-triangular matrices have integer entries on the diagonal.



# The filters $\mathcal{F}$

A filter  $F$  is a surjective map  $F : X \rightarrow [1, 2, \dots, k]$ ,  $k \leq n$ , such that if  $\alpha_i \preceq \alpha_j$  then  $F(\alpha_i) \leq F(\alpha_j)$ . (For  $k = n$  a filter is the same as a linear order.) For  $b_1, \dots, b_r$  being integers summing up to  $n$  we denote by  $\mathcal{F}_{b_1, \dots, b_r}$  the set of all filters  $F : X \rightarrow [1, 2, \dots, r]$  such that  $|F^{-1}(i)| = b_i$  for all  $i = 1, \dots, r$ .

Let  $P$  be a linear order on  $X$  and  $F$  a filter on  $X$ . We define a linear order  $Q(P, F)$  by the relations:

- 1 for  $\alpha_i, \alpha_j$  in the same strata, i.e.  $F(\alpha_i) = F(\alpha_j)$  we have  $Q(\alpha_i) < Q(\alpha_j)$  iff  $P(\alpha_i) < P(\alpha_j)$ .
- 2 for  $\alpha_i, \alpha_j$  in different stratas we have  $Q(\alpha_i) < Q(\alpha_j)$  iff  $F(\alpha_i) < F(\alpha_j)$ .

We define the matrix  $M_F$  by

$$(M_F)_{PQ} = \begin{cases} 1 & \text{if } Q = Q(P, F) \\ 0 & \text{in all other cases} \end{cases} .$$

In particular, the matrix  $M_F$  has exactly one non-zero entry in every row.

# The filter semigroup and the face semigroup

The central real hyperplane arrangement  $A_n$  (braid arrangement) of hyperplanes  $\{H_{ij} : 1 \leq i < j \leq n\}$  in  $\mathbb{R}^n$  defined by

$$H_{ij} = \{(x_1, \dots, x_n) : x_i = x_j\}$$

A chamber is an open connected component of

$$\mathbb{R}^n \setminus \{\cup H_{ij}\}$$

A *cone* is a union of closures of chambers, which is *convex*.

$\mathfrak{D}(n)$  is the set of all different cones.

Let a poset  $X$  of  $n$  elements be given, with a binary relation  $\preceq$ . To every pair  $i, j \in X$  such that

$$i \preceq j$$

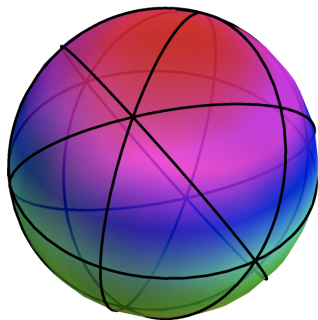
there corresponds a half-space  $K_{ij} = \{x_i \leq x_j\} \subset \mathbb{R}^n$ . Consider the cone

$$A(X, \preceq) = \left\{ \bigcap_{i,j:i \preceq j} K_{ij} \right\} \in \mathfrak{D}(n)$$

where the intersection is taken over all pairs  $i, j$  such that  $i \preceq j$ .

The above defined correspondence  $(X, \preceq) \rightarrow A(X, \preceq)$  is a one-to-one correspondence between the set of all partial orders on  $\{1, 2, \dots, n\}$  and the set of all (convex) cones  $\mathfrak{D}(n)$ .

$$n = 4$$



**Figure:** The central real hyperplane arrangement  $A_4$  in  $\mathbb{R}^4$ , projected to  $\mathbb{R}^3$  along the line  $x = y = z = t$  and intersected with the sphere  $\mathbb{S}^2 \subset \mathbb{R}^3$ .

$$n = 4$$

A partition of  $\mathbb{S}^2$  into 24 equal triangles  $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$ . Convex unions are: the sphere, the hemisphere, the moon, an elementary triangle (e-triangle), a pair of e-triangles with a common side, a triangle made from three e-triangles, a 'square' formed by four e-triangles, a triangle made from a 'square' and a fifth adjacent e-triangle, a triangle formed by six e-triangles with a common  $\frac{\pi}{3}$ -vertex. Their numbers are 1, 12, 60, 24, 36, 48, 6, 24, 8, totally 219. This is precisely the number of partial orders on the set of four distinct elements.



Let  $f', f''$  be two faces in  $A(X) = A(X, \preceq)$ . Define the face  $f = f''(f') \in A(X)$  – or the *face-product*  $f''f'$ :

Choose points  $x' \in f', x'' \in f''$  in general position.

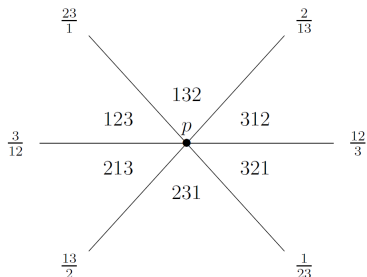
Let  $s_{x'x''} : [0, 1] \rightarrow \mathbb{R}^n$  be a linear segment,  $s_{x'x''}(0) = x', s_{x'x''}(1) = x''$ .

Consider the face  $f \in A(X)$  containing all the points  $s_{x'x''}(1 - \varepsilon)$  of the segment for  $\varepsilon > 0$  small enough.

By definition,  $f''(f') = f$ .

# Face-product

This is the picture of the projection of the braid arrangement in  $\mathbb{R}^3$  to the plane orthogonal to the main diagonal:



If  $f''$  is a chamber then  $f''f' = f''$ .

If  $f''$  is a chamber then  $f'f''$  is also a chamber, so the faces are acting on chambers.

The face-product is associative.

For every choice of faces  $f, g, h \in A(X, \preceq)$  we have

$$f(gh) = (fg)h,$$

The semigroup  $A(X, \preccurlyeq)$  is a *left-regular band* :

$$ff = f,$$

$$fgf = fg.$$

The semigroup  $A(X, \preccurlyeq)$  defines back the poset  $X$ .

# Filters and Faces

Let  $F$  be a filter on  $X$  of rank  $k$ , i.e. a surjective map  $F : X \rightarrow \{1, \dots, k\}$ , preserving the partial order, and let

$$\{b_1, \dots, b_{j_1}\}, \{b_{j_1+1}, \dots, b_{j_2}\}, \dots, \{b_{j_{k-1}+1}, \dots, b_{j_k}\} \subset X$$

be its 'floors':

$$\{b_{j_{r-1}+1}, \dots, b_{j_r}\} = F^{-1}(r), \quad r = 1, \dots, k.$$

Consider the face  $f_F \in A(X, \preceq)$ , defined by the equations

$$x_{b_{j_{r-1}+1}} = \dots = x_{b_{j_r}}, \quad r = 1, \dots, k$$

and inequalities

$$x_{b_{j_1}} < x_{b_{j_2}} < \dots < x_{b_{j_k}}.$$

This is a one-to-one correspondence between faces and filters. The filters of the highest rank  $n$ , i.e. the linear extensions of  $\preceq$ , correspond to the chambers.

The corresponding filter-product looks as follows. For  $F', F''$  being two filters of  $X$ , the filter  $F = F''F'$  on  $X$  is uniquely defined by the following properties:

- For  $u, v$  with  $F''(u) < F''(v)$  we have  $F(u) < F(v)$ .
- For  $u, v$  with  $F''(u) = F''(v)$  we have  $F(u) < F(v)$  iff  $F'(u) < F'(v)$ .

Let  $F$  be a filter on  $X$ , and  $P$  is some filter of rank  $n$ , i.e. a linear order on  $X$ . Then the filter  $FP$  is again a filter of rank  $n$ . Consider the square matrix  $M_F^X = \left\| (M_F^X)_{P,Q} \right\|$  where  $P, Q$  are linear orders on  $X$  :

$$(M_F^X)_{P,Q} = \begin{cases} 1 & \text{if } Q = FP \\ 0 & \text{if } Q \neq FP \end{cases} .$$

The operators  $M_F^X$  play a central role in our proof.



# The filters $F$ – back to pedestals

Let us rewrite our pedestal matrix  $M_X$  as the sum over all monomials,

$$M_X = \sum_{r=1}^n \sum_{\substack{a_1, \dots, a_r \geq 1 \\ a_1 + \dots + a_r = n}} x_1^{a_1} \dots x_r^{a_r} B_{a_1, \dots, a_r},$$

where the entries of each matrix  $B_{a_1, \dots, a_r}$  are 0 or 1.

# The filters $F$

We claim that if  $B_{a_1, \dots, a_r} \neq 0$  then the following inclusion-exclusion identity holds:

$$B_{a_1, \dots, a_r} = \sum_{F \in \mathcal{F}_{a_1, \dots, a_r}} M_F - \left[ \sum_{\substack{F \in \mathcal{F}_{a_1+a_2, a_3, \dots, a_r} \cup \\ \cup \mathcal{F}_{a_1, a_2+a_3, \dots, a_r} \cup \dots}} M_F \right] \quad (1)$$
$$+ \left[ \sum_{\substack{F \in \mathcal{F}_{a_1+a_2+a_3, a_4, \dots, a_r} \cup \\ \cup \mathcal{F}_{a_1+a_2, a_3+a_4, \dots, a_r} \cup \dots}} M_F \right] - \dots$$

where the sums are taken over all possible mergers of neighboring indices  $a_i$ , and the signs are  $(-1)^{\#\text{mergers}}$ .

# The filters $F$

Indeed, an order  $Q$  from the row  $P$  from the lhs, agrees with  $P$  over the first  $a_1 - 1$  locations, then disagrees once, then agrees again over next  $a_2 - 1$  locations, then disagrees once again, etc. An order  $Q$  from the row  $P$  which appears in the rhs and corresponds to the first sum, agrees with  $P$  over the first  $a_1 - 1$  locations, then it **agrees or disagrees** once, then agrees again over next  $a_2 - 1$  locations, then **agrees or disagrees** once again, etc. Therefore we have to remove all these  $Q$ -s which agrees with  $P$  over the first  $a_1 - 1$  locations, then **agrees** once again, then agrees also over next  $a_2 - 1$  locations, etc.

Our matrices  $M_{F,X}$  are of the size  $|\text{Tot}_X| \times |\text{Tot}_X|$ . Let us now abolish all order relations on  $X$ , getting the poset  $\bar{X}$  with  $|\text{Tot}_{\bar{X}}| = n!$ . Of course,  $M_{F,X}$  is a submatrix of  $M_{F,\bar{X}}$ . Let it be an upper-left submatrix.

To the right of it all matrix elements of  $M_{F,\bar{X}}$  are zero, and so  $M_{F,X}$  is a block of  $M_{F,\bar{X}}$ . Indeed, each row of  $M_{F,\bar{X}}$  has exactly one 1, and the rest are 0-s. But each row of  $M_{F,X}$  already has one 1. So it is sufficient to know that the spectrum of  $M_{F,\bar{X}}$  consists of integers.

We will deal only with 'totally unordered' poset  $\bar{X}$ .

Let us consider an even bigger matrices,  $M_{F,T}$ , of size  $2^{n(n-1)/2}$ . Here  $T$  stays for *tournaments* between  $n$  entries.

A tournament is an assignment of the order  $\preceq$  to every pair  $i \neq j$  of the elements of the set  $[1, \dots, n]$ , independently for each pair.

If we have a tournament  $\preccurlyeq$  and a filter  $F$  then we define a new tournament  $\preccurlyeq_F$  by the rule:

- 1 If  $F(i) = F(j)$  then  $i \preccurlyeq_F j$  iff  $i \preccurlyeq j$ ,
- 2 If  $F(i) < F(j)$  then  $i \preccurlyeq_F j$ .

Any linear order defines a tournament in an obvious way, so our matrices  $M_{F,X}$  are blocks of  $M_{F,T}$ -s, and it is sufficient to study  $M_{F,T}$ -s.

The key observation:  $M_{F,T}$  is a tensor product of  $n(n-1)/2$  two-by-two matrices, corresponding to all pairs  $(i,j)$ , since the orders  $\preceq$  can be assigned to the pairs independently.

Since the tensor product of upper triangular matrices is upper triangular, it is sufficient to check our claim just for the filters and tournaments in the case  $n = 2$ .

The three possible  $M_{F,T}$ -s in this case are

$$M_1 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, M_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$



Conjugating them by the discrete Fourier transform matrix

$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  brings them to the triple of upper triangular

matrices:  $UM_1U^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $UM_2U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , and

$UM_3U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ . That finishes the proof.

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